Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 377–386
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

The soliton equations associated with the affine Kac–Moody Lie algebra $G_2^{(1)}$

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Received 1 October 2007; accepted 26 November 2007
Available online 24 January 2008

Abstract

We construct in an explicit way the soliton equations corresponding to the affine Kac–Moody Lie algebra $G_2^{(1)}$ together with their bi-Hamiltonian structure. Moreover the Riccati equation satisfied by the generating function of the commuting Hamiltonian densities is also deduced. Finally we describe a way to deduce the bi-Hamiltonian equations directly in terms of this latter function.

Keywords: Bi-Hamiltonian integrable systems; W-algebras

1. Introduction

One of the most fascinating discoveries of the last few decades is surely the deep and fundamental link between the affine Kac–Moody Lie algebras (and their groups as well), and the soliton equations. This relation, first studied and described from different points of view in a sequence of seminal papers by Sato [16,17], Date, Jimbo, Kashiwara and Miwa [10], Hirota [12], Drinfeld and Sokolov [11], and Kac and Wakimoto [14], has inspired almost innumerable further investigations and generalizations (see for example the quite interesting papers of Burroughs, de Groot, Hollowood, Miramontes [1,2]). Nevertheless, as far as we know, it seems that no explicit description of the hierarchy corresponding in the scheme of Drinfeld and Sokolov to the affine Kac–Moody Lie algebra $G_2^{(1)}$ (even of the first non-trivial equations) can be found in the literature, a fact probably related to the size of the standard realization of G_2 (namely in 7×7 matrices). The aim of this work is to fill this gap and to show how the bi-Hamiltonian formulation of the Drinfeld–Sokolov reduction [7,8,3] makes the computations involved more reasonable. The main ingredient of our construction will be indeed the technique of the transversal submanifold, which can be implemented only in the bi-Hamiltonian reduction theory, and which reduces the number of free variables involved in the computations. The same technique provides also a way to construct a Riccati type equation for the formal Laurent series for the conserved

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.11.012

quantities of the corresponding integrable system. Since this equation at least in principle may be iteratively solved in a pure algebraic way, the bi-Hamiltonian technique offers a computational way to construct the whole hierarchy. Moreover, what happens in the case of the affine Lie algebras $A_n^{(1)}$ suggests that there could exist a way to obtain directly the equations of the hierarchy, starting from such conserved quantities, without referring directly to the underlying bi-Hamiltonian structure.

The paper is organized as follows. In the first section we perform the bi-Hamiltonian reduction of the Drinfeld–Sokolov hierarchy defined on the affine Kac–Moody Lie algebra $G_2^{(1)}$, obtaining the reduced bi-Hamiltonian structures and the first equations of the hierarchy as well. In the second and last sections we explain and perform the so called Frobenius technique for the same algebra, obtaining a so called Riccati equation satisfied by the generating function of the conserved densities. Finally we shall show how the knowledge of this function is enough for constructing the entire hierarchy of bi-Hamiltonian equations.

2. The bi-Hamiltonian reduction theory of the Lie algebra $G_2^{(1)}$

The aim of this first section is to obtain the bi-Hamiltonian structure of the soliton equation associated with the Kac–Moody affine Lie algebra $G_2^{(1)}$ in the Drinfeld–Sokolov theory by performing a bi-Hamiltonian reduction process.

For the convenience of the reader, let us start by recalling the main facts of the bi-Hamiltonian reduction theory, referring the reader for more details to the papers [7,8,3] where this theory was first developed. A bi-Hamiltonian manifold $\mathcal{M}$ is a manifold equipped with two compatible Poisson structures, i.e., two Poisson tensors P_0 and P_1 such that the pencil $P_\lambda = P_1 - \lambda P_0$ is a Poisson tensor for any $\lambda \in \mathbb{C}$. Let us fix a symplectic submanifold $\mathcal{S}$ of P_0 and consider the distribution $D = P_1 \mathrm{Ker}(P_0)$; then the bi-Hamiltonian structure of $\mathcal{M}$, provided that the quotient space $\mathcal{N} = \mathcal{S}/E$, $E = D \cap T\mathcal{S}$ is a manifold, can be reduced to one on $\mathcal{N}$ ([7] Prop. 1.1).

To construct the reduced Poisson pencil $P_\lambda^{\mathcal{N}}$ from the Poisson pencil P_λ on $\mathcal{M}$ we have to perform the following steps [5]:

1. For any 1-form α on $\mathcal{N}$ we consider the 1-form $\pi^*\alpha$ on $\mathcal{S}$, which obviously belongs to the annihilator E^0 of E in $T^*\mathcal{S}$.
2. We construct a 1-form β on $\mathcal{M}$ which belongs to the annihilator D^0 of D and satisfies
$$i_{\mathcal{S}}^*\beta = \pi^*\alpha \tag{2.1}$$
(i.e., a lifting of α).
3. We compute the vector field $P_\lambda\beta$, which turns out (see [5] Lemma 2.2) to be tangent to $\mathcal{S}$.
4. We project $P_\lambda\beta$ on N:
$$(P_\lambda^{\mathcal{N}})_{\pi(s)}\alpha = \pi_*(P_\lambda)_s\beta.$$

We shall not compute in the next section the reduced bi-Hamiltonian structure related to the affine Kac–Moody Lie algebra $G_2^{(1)}$ using directly the above cited theorem but rather implementing the technique of the transversal submanifold [8,5] in order to avoid most of the computations involved.

A transversal submanifold to the distribution E is a submanifold $\mathcal{Q}$ of $\mathcal{S}$, which intersects every one of the integral leaves of the distribution E in one and only one point. This condition implies the following relations on the tangent space:
$$T_q\mathcal{S} = T_q\mathcal{S} \oplus E_q \quad \forall q \in \mathcal{Q}. \tag{2.2}$$
The importance of the knowledge of a transversal submanifold lies in the following theorem proved in [7,3]:

Theorem 2.1. *Let $\mathcal{Q}$ be a transversal submanifold of $\mathcal{S}$ with respect the distribution E. Then $\mathcal{Q}$ is a bi-Hamiltonian manifold isomorphic to the bi-Hamiltonian manifold $\mathcal{N}$ and the corresponding reduced Poisson pair is given by*
$$\left(P_i^{\mathcal{Q}}\right)_q \alpha = \Pi_*(P_i)_q\tilde{\alpha} \quad i = 0, 1 \tag{2.3}$$

where $q \in \mathcal{Q}$, $\alpha \in T_q^*\mathcal{Q}$, $\Pi_* : T_q\mathcal{S} \to T_q\mathcal{Q}$ *is the projection with respect the decomposition* (2.2) *and* $\tilde{\alpha} \in T_q^*\mathcal{M}$ *satisfies the conditions*

$$\tilde{\alpha}_{|D_q} = 0 \qquad \tilde{\alpha}_{|T_q\mathcal{Q}} = \alpha. \tag{2.4}$$

Actually for our purposes the hypothesis of this theorem may be slightly relaxed by considering a submanifold $\mathcal{Q}$ which is only locally transversal (i.e., it satisfies only the weaker condition (2.2)); in this case of course $\mathcal{Q}$ and $\mathcal{N}$ could be only locally isomorphic (see [15] for more details).

The bi-Hamiltonian manifolds which we are interested in are the bi-Hamiltonian manifolds naturally defined on the affine Kac–Moody Lie algebras. An affine non-twisted Lie algebra $\widehat{\mathfrak{g}}$ can be realized as a semidirect product of the central extensions of a loop algebra of a simple finite dimensional Lie algebra $\mathfrak{g}$ and a derivation d [13]:

$$\widehat{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g}) \oplus \mathbb{C}d \oplus \mathbb{C}c.$$

Then the Lie bracket of two (typical) elements in $\widehat{\mathfrak{g}}$ of the type

$$X = f \otimes x^n + \mu_1 c + \nu_1 d, \qquad Y = g \otimes x^m + \mu_2 c + \nu_2 d$$

with $f, g \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$, $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{C}$ is

$$[X, Y] = [f, g] \otimes x^{n+m} + (f, g)c\delta_{n+m,0} - n\nu_2 f \otimes x^n + m\nu_1 g \otimes x^m \tag{2.5}$$

where $[\cdot,\cdot]$ is the Lie bracket in $\mathfrak{g}$, and $(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}$ (and finally δ is the usual Kronecker delta). In what follows the derivation d will not play any role. As $\widehat{\mathfrak{g}}$ is an affine (infinite dimensional) manifold we may identify it with its tangent space at any point. Moreover using the non-degenerate form

$$\langle (V_1, a), (V_2, b)\rangle = \int_{S^1} (V_1(x), V_2(x))\mathrm{d}x + ab \tag{2.6}$$

we may identify at any point S the tangent space with the corresponding cotangent space $T_S\mathcal{M} = T_S^*\mathcal{M}$. Using these identifications we can write the canonical Lie Poisson tensor of $\widehat{\mathfrak{g}}$ as

$$P_{(S,c)}(V) = c\partial_x V - [S, V]. \tag{2.7}$$

It can be easily shown that this Poisson tensor is compatible with the constant Poisson tensor obtained by freezing the tensor (2.7) at any point of $\mathcal{M}$. In particular the hierarchies of Drinfeld and Sokolov turn out to be bi-Hamiltonian with respect to the bi-Hamiltonian pair P_1, P_0 where P_1 is the canonical Poisson tensor (2.7) and P_0 is the constant Poisson tensor

$$(P_0)_{(S,c)}(V) = [A, V]. \tag{2.8}$$

where A is the constant function of $C^\infty(S^1, \mathfrak{g})$ whose value is the element of minimal weight in $\mathfrak{g}$.

3. The reduction process

In this section, following [8], we perform the bi-Hamiltonian reduction of the exceptional Lie algebra $G_2^{(1)}$. It is a rank 2 simple Lie algebra whose Cartan matrix is $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. A possible Weyl basis is

$$\begin{aligned} &H_1 = d_{22} - d_{33} + d_{55} - d_{66} \qquad H_2 = d_{11} - d_{22} + 2d_{33} - 2d_{55} + d_{66} - d_{77} \\ &E_1 = d_{23} + d_{56} \qquad E_2 = d_{12} + d_{34} + 2d_{45} + d_{67} \\ &F_1 = d_{32} + d_{65} \qquad F_2 = d_{21} + 2d_{43} + d_{54} + d_{76} \end{aligned} \tag{3.1}$$

where d_{ij} is a 7×7 matrix with 1 in the ij position and zeros elsewhere.

Thus the elements of the algebra have the form

$$v = \begin{bmatrix} h_2 & e_2 & -e_3 & 2\,e_4 & -6\,e_5 & 6\,e_6 & 0 \\ f_2 & h_1-h_2 & e_1 & e_3 & -2\,e_4 & 0 & 6\,e_6 \\ f_3 & f_1 & -h_1+2\,h_2 & e_2 & 0 & -2\,e_4 & 6\,e_5 \\ 4\,f_4 & -2\,f_3 & 2\,f_2 & 0 & 2\,e_2 & -2\,e_3 & 4\,e_4 \\ 6\,f_5 & -2\,f_4 & 0 & f_2 & h_1-2\,h_2 & e_1 & e_3 \\ 6\,f_6 & 0 & -2\,f_4 & f_3 & f_1 & -h_1+h_2 & e_2 \\ 0 & 6\,f_6 & -6\,f_5 & 2\,f_4 & -f_3 & f_2 & -h_2 \end{bmatrix}.$$

As already noted, on $G_2^{(1)}$ there is defined a bi-Hamiltonian structure given by the canonical Lie Poisson tensor and by the tensor (2.8) where in the present case the element of minimal weight is $A = F_6$.

To perform the Marsden–Ratiu reduction process on such bi-Hamiltonian structure we need to compute first a symplectic leaf $\mathcal{S}$ of P_0 and second the distribution E at the points of $\mathcal{S}$. As proved in [8] the symplectic leaves of the constant Poisson tensor are affine subspaces modelled on the subspace of $G_2^{(1)}$ orthogonal to the isotropic algebra of the element A. Following Drinfeld and Sokolov let us choose that passing through the point $B = E_1 + E_2$,

$$S = B + h(t)(2H_1 + H_2) + f_1(t)F_1 + f_3(t)F_3 + f_4(t)F_4 + f_5(t)F_5 + f_6(t)F_6.$$

Then the (constant) distribution E evaluated at the points of $\mathcal{S}$ is

$$\begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 & 0 & 0 \\ 2\,t_4 & -2\,t_2 & 0 & 0 & 0 & 0 & 0 \\ t_5 & -t_4 & 0 & 0 & 0 & 0 & 0 \\ t_6 & 0 & -t_4 & t_2 & t_3 & -t_1 & 0 \\ 0 & t_6 & -t_5 & t_4 & -t_2 & 0 & -t_1 \end{bmatrix}.$$

Luckily, we may apply Theorem 2.1, since the submanifold $\mathcal{Q}$ of $\mathcal{S}$

$$\mathcal{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & u_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6\,u_1 & 0 & 0 & 0 & u_0 & 0 & 1 \\ 0 & 6\,u_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is transversal to E.

The actually computation of the explicit form of the reduced Poisson pencil as observed in [4] boils down to finding (given a 1-form $v = (v_0, v_1) \in T^*(\mathcal{Q})$) a section $V(v)$ in $i_{\mathcal{Q}}^*(T^*\mathcal{M})$ (where $i_{\mathcal{Q}} : \mathcal{Q} \hookrightarrow \mathcal{M}$ is the canonical inclusion) such that $P_\lambda\, V(v) \in T\,Q$. This implies that the entries of $V(v)$ are polynomial functions of the elements $(8\,e_1, 288\,e_6)$ and that the reduced Poisson pencil,

$$\frac{\mathrm{d}q}{\mathrm{d}t_\lambda} = (P_\lambda^{\mathcal{Q}})_q v = V(v)_x + [V(v) + \lambda A, q],$$

becomes

$$\begin{aligned} \frac{\mathrm{d}u_0}{\mathrm{d}t_\lambda} = {} & \left(\frac{7}{32}u_0' v_1^{(4)} + \frac{3}{4}u_1 v_1' - \frac{1}{8}\left(u_0'\right)^2 v_1' + \frac{11}{32}u_0'' v_1''' + \frac{5}{32}u_0^{(4)} v_1' + \frac{5}{16}u_0''' v_1'' + \frac{1}{16}u_0 v_1^{(5)} \right. \\ & + \frac{5}{8}u_1' v_1 - \frac{1}{32}u_0^2 v_1''' + \frac{1}{32}u_0^{(5)} v_1 + \frac{1}{4}u_0 v_0' + \frac{1}{8}u_0' v_0 - \frac{7}{4}v_0''' - \frac{1}{32}v_1^{(7)} - \frac{1}{8}u_0 u_0'' v_1' \\ & \left. - \frac{5}{32}u_0 u_0' v_1'' - \frac{3}{32}u_0' u_0'' v_1 - \frac{1}{32}u_0 u_0''' v_1 \right) - \lambda \frac{3}{4} v_1' \end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}u_1}{\mathrm{d}t_\lambda} = {} & \Big(\frac{1}{96}u_0'v_1^{(8)} - \frac{7}{144}(u_0''')^2v_1' - \frac{1}{1728}u_0^4v_1''' + \frac{1}{1728}u_0^{(9)}v_1 - \frac{1}{32}u_0^2v_0''' + \frac{3}{4}u_1v_0' \\
& + \frac{1}{72}u_1v_1^{(5)} + \frac{47}{576}u_0^{(4)}v_1^{(5)} + \frac{13}{144}u_1''v_1''' + \frac{5}{288}u_0^2u_0'v_1^{(4)} + \frac{7}{864}u_0u_0'u_0^{(4)}v_1 \\
& - \frac{61}{576}(u_0'')^2v_1''' + \frac{1}{432}u_0v_1^{(9)} + \frac{5}{96}u_1^{(4)}v_1' + \frac{1}{8}u_1'v_0 - \frac{1}{1728}v_1^{(11)} - \frac{23}{576}u_0u_0^{(5)}v_1'' \\
& + \frac{1}{16}u_0v_0^{(5)} + \frac{29}{288}u_1'''v_1'' + \frac{85}{1728}u_0^{(6)}v_1''' + \frac{1}{48}u_0^{(7)}v_1'' + \frac{3}{32}u_0'v_0^{(4)} - \frac{1}{24}u_1u_0''v_1' \\
& - \frac{1}{24}u_0u_1'v_1'' - \frac{1}{36}u_1u_0v_1''' - \frac{1}{24}u_1u_0'v_1'' + \frac{23}{864}u_0''u_0^2v_1''' + \frac{29}{864}u_0(u_0')^2v_1''' \\
& - \frac{17}{96}u_0''u_0'''v_1'' - \frac{5}{64}u_0''u_0^{(4)}v_1' - \frac{1}{72}u_0''u_0^{(5)}v_1 - \frac{7}{1728}u_0^2(u_0')^2v_1' - \frac{67}{432}u_0'u_0'''v_1''' \\
& - \frac{1}{48}u_0^{(4)}u_0'''v_1 - \frac{1}{32}u_0u_0'v_0'' + \frac{1}{144}u_0^2u_1'v_1 + \frac{1}{72}u_0^2u_1v_1' - \frac{1}{432}u_0^3u_0''v_1' - \frac{1}{1728}u_0^3u_0'''v_1 \\
& - \frac{1}{288}u_0^3u_0'v_1'' - \frac{1}{1728}u_0(u_0')^3v_1 - \frac{11}{864}u_0u_0^{(6)}v_1' - \frac{1}{576}u_0u_0^{(7)}v_1 - \frac{5}{48}u_0'u_0^{(4)}v_1'' \\
& - \frac{11}{288}u_0'u_0^{(5)}v_1' - \frac{5}{864}u_0'u_0^{(6)}v_1 - \frac{61}{864}u_0u_0^{(4)}v_1''' + \frac{3}{32}u_0''v_0''' + \frac{35}{576}u_0'''v_1^{(6)} + \frac{1}{64}(u_0')^3v_1'' \\
& - \frac{1}{32}v_0^{(7)} + \frac{35}{864}u_0u_0'u_0'''v_1' - \frac{1}{18}u_0u_0''v_1^{(5)} - \frac{1}{36}u_0'u_1'v_1' + \frac{1}{72}u_0u_0'u_1v_1 \\
& + \frac{1}{32}u_0'''v_0'' + \frac{5}{144}u_1'v_1^{(4)} - \frac{1}{288}u_0^2v_1^{(7)} + \frac{1}{72}u_0u_0''u_0'''v_1 - \frac{1}{432}u_0^2u_0'u_0''v_1 \\
& + \frac{11}{144}u_0^{(5)}v_1^{(4)} + \frac{11}{144}u_0u_0'u_0''v_1'' - \frac{5}{64}u_0u_0'''v_1^{(4)} - \frac{5}{288}u_0u_1'''v_1 - \frac{5}{288}u_0'u_1''v_1 \\
& - \frac{7}{144}u_0u_1''v_1' - \frac{1}{72}u_1'u_0''v_1 + \frac{19}{576}(u_0')^2u_0''v_1' + \frac{13}{576}u_0^2u_0'''v_1'' + \frac{17}{1728}u_0^2u_0^{(4)}v_1' \\
& + \frac{1}{96}u_0'(u_0'')^2v_1 + \frac{1}{36}u_0(u_0'')^2v_1' + \frac{13}{1728}u_0'''(u_0')^2v_1 + \frac{1}{576}u_0^2u_0^{(5)}v_1 - \frac{1}{72}u_1u_0'''v_1 \\
& - \frac{7}{288}u_0u_0'v_1^{(6)} - \frac{5}{36}u_0'u_0''v_1^{(4)} + \frac{1}{96}u_1^{(5)}v_1 + \frac{1}{432}u_0^3v_1^{(5)} + \frac{1}{192}u_0^{(8)}v_1' \\
& + \frac{1}{32}u_0''v_1^{(7)} - \frac{7}{192}(u_0')^2v_1^{(5)}\Big) - \lambda\Big(-\frac{1}{72}u_0'''v_1 + \frac{1}{72}u_0^2v_1' + \frac{1}{72}v_1^{(5)} - \frac{1}{24}u_0''v_1' \\
& + \frac{1}{72}u_0u_0'v_1 - \frac{1}{24}u_0'v_1'' - \frac{1}{36}u_0v_1''' + \frac{3}{4}v_0'\Big)
\end{aligned}
$$

where the prime indicates the space derivative. Having the reduced bi-Hamiltonian structure we are now able to write explicitly the first non-trivial flows of the hierarchy [7]. Since the Casimirs of P_0 are given by the functionals

$$H_0 = \int_{S^1} \mathrm{d}x\, u_0 \quad \text{and} \quad H_1 = \int_{S^1} \mathrm{d}x\, u_0''u_0 - \frac{u_0^3}{3} + 108u_1 \tag{3.2}$$

we obtain

$$u_{0,t_0} = \frac{1}{8}u_{0x} \tag{3.3}$$

$$u_{1,t_0} = \frac{1}{8}u_{1x} \tag{3.4}$$

and

$$u_{0,t_1} = -\frac{1}{864}(u_0^{(5)} + 5u_0'u_0^2 - 5u_0'''u_0 - 5u_0''u_0' - 540u_1) \tag{3.5}$$

$$u_{1,t_1} = -\frac{1}{864}(-9u_1^{(5)} + 15u_1'''u_0 + 15u_1''u_0' + 10u_1'u_0'' - 5u_1'u_0^2) \tag{3.6}$$

4. The Frobenius technique

In the previous section we found the bi-Hamiltonian structure of the soliton equation associated with the affine Kac–Moody Lie algebra $G_2^{(1)}$ together with its first non-trivial equations. This is of course far from being the same as providing a way to actually compute all the soliton equations of the hierarchy. In the setting of the bi-Hamiltonian theory this second important problem is tackled by looking for Casimirs of the Poisson pencil $P_\lambda = P_1 - \lambda P_0$, i.e., solutions of the equations

$$V_x + [V, S + \lambda A] = 0 \qquad s \in \mathcal{S} \tag{4.1}$$

which are formal Laurent series $V(\lambda) = \sum_{k=-1}^{\infty} V_k \lambda^{-k}$ whose coefficients are 1-forms defined at least at the points of $\mathcal{S}$ which are exact when restricted to $\mathcal{S}$. Indeed once such a solution is found the vector fields of the hierarchy can be written in the bi-Hamiltonian form $X_k = P_0 V_k = P_1 V_{k-1}$.

This latter task is unfortunately usually a very tough problem, which can be solved by using the generalization of the dressing method of Zakharov and Shabat proposed by Drinfeld and Sokolov [11]. Unfortunately, exactly as happens for the Drinfeld–Sokolov reduction for the Lie algebra $G_2^{(1)}$, the computations involved in deriving the explicit expression for the bi-Hamiltonian fields of the hierarchy are very complicated. The aim of this last section is to show how the so called Frobenius technique [6] provides something of a shortcut in the Drinfeld–Sokolov procedure.

More precisely, this technique will give a way to compute algebraically by a recursive procedure the conserved densities of the hierarchy and therefore the corresponding (maybe without passing through the Poisson tensors) bi-Hamiltonian vector fields as well. However, implementing such a technique requires us to give up the pure geometrical description of the hierarchy of the first section in order to consider also the minimal true loop module $C^\infty(S^1, \mathbb{R}^7)$ of $G_2^{(1)}$ [9], together with its geometrical dual space and the set of its linear automorphisms as well.

The starting point of the theory is indeed to observe [4] that $V \in T_S^*\mathcal{S}$ solves (4.1) at the point $S \in (\mathcal{S}, c = 1)$ if and only if it commutes viewed as linear operator in $\mathrm{End}(C^\infty(S^1, \mathbb{C}^7))$ (up to the canonical identification explained in the previous section) with the linear differential operator $-c\partial_x + S + \lambda A$. Although this latter task seems not really easier than the first one, it suggests a way (using the action of the affine Lie group $\widetilde{\mathfrak{G}}$ on the representation space $C^\infty(S^1, \mathbb{R}^7)$) to obtain directly the equations of the hierarchy together with their Hamiltonians. Following what was suggested by Drinfeld and Sokolov we can find the elements V commuting with $-c\partial_x + S + \lambda A$ using the observation that the element $B + \lambda A$ is a regular element and therefore its isotropic subalgebra $\mathfrak{g}_{B+\lambda A}$ is a Heisenberg subalgebra $\mathfrak{H}$ of $\widetilde{\mathfrak{g}}$ spanned (up to the central charge) in our representation by the matrices $\Lambda^{6n+1} = \left(\frac{\lambda}{24}\right)^n (B + \frac{\lambda}{24}A)$, $\Lambda^{6n-1} = \left(\frac{\lambda}{24}\right)^{n-1} (B + \frac{\lambda}{24}A)^5$ with $n \in \mathbb{Z}$. (For the sake of simplicity, from now on, we rescale $\frac{\lambda}{24} \to \lambda$.) From this fact Drinfeld and Sokolov indeed prove the

Proposition 4.1. *For any operator of the form $-\partial_x + S + \lambda A$ with $s \in \mathcal{S}$ there exists a element T in $\widetilde{\mathfrak{G}}$ such that*

$$T(-\partial_x + S + \lambda A)T^{-1} = \partial_x + (B + \lambda A) + H, \quad H \in \mathfrak{H}. \tag{4.2}$$

Therefore the set of the elements in $\widetilde{\mathfrak{g}}$ commuting with $-\partial_x + S + \lambda A$ is given by $T^{-1}\mathfrak{H}T$.

The knowledge of a such a T allows us to compute explicitly for any choice of an element in $\mathfrak{H}$ the corresponding hierarchy of vector fields together with their Hamiltonians.

Proposition 4.2. *Let $C = \sum_{j=\pm 1 \bmod (6)} c_j \Lambda^{-j}$ with $c_j \in \mathbb{C}$ be an element in $\mathfrak{H}$; then:*

1. *the element $V_C = T^{-1}CT$ solves Eq. (4.1);*
2. *its Hamiltonian on $\mathcal{S}$ is the function $H_C = \langle J, C\rangle$ where J is defined by the relation*

$$J = T(S + \lambda A)T^{-1} + T_x T^{-1}; \tag{4.3}$$

3. *in particular if C has the form $C = \Lambda^j$, $j = \pm 1 \bmod (6)$ (say $j = 6n \pm 1$), then V_C and H_C, simply denoted as V^j, have the Laurent expansion*

$$V^j = \lambda^n \sum_{p \geq -2} \frac{1}{\lambda^{p+1}} V_{6p\pm 1}. \tag{4.4}$$

Proof. 1. It was already proved above.

2. Using Eq. (4.3) we can rewrite Eq. (4.2) in the form $T(-\partial_x + S + \lambda A)T^{-1} = -\partial_x + J$ showing that J commutes with C; then

$$\frac{\mathrm{d}}{\mathrm{d}t} H_C = \langle \dot{J}, C\rangle = \left\langle T\dot{S}T^{-1} + \left[\dot{T}T^{-1}, J\right], C\right\rangle$$

but since C commutes with J we get

$$\frac{\mathrm{d}}{\mathrm{d}t} H_C = \langle T\dot{S}T^{-1}, C\rangle = \langle \dot{S}, T^{-1}CT\rangle = \langle \dot{S}, V_C\rangle.$$

3. Eq. (4.4) follows for $j = 6n + 1$ from the identity

$$\begin{aligned}\mathrm{res}(\lambda^{p-n}V^j) &= \mathrm{res}(\lambda^{p-n}T\Lambda^jT^{-1}) = \mathrm{res}(\lambda^{p-n}T\lambda^n\Lambda T^{-1})\\ &= \mathrm{res}(T\lambda^p\Lambda T^{-1}) = \mathrm{res}(T\Lambda^{6p+1}T^{-1}) = \mathrm{res}(V^{6p+1})\end{aligned}$$

while similar computations show that if $j = 6n - 1$ then $\mathrm{res}(\lambda^{p-n}V^j) = \mathrm{res}(V^{6p-1})$. □

As already pointed out, the actual computation of the element T (which by the way provides also the vector fields of the hierarchy) is in our case quite complicated. To avoid such computations let us first solve the related problem of finding the eigenvalues of the operator $-\partial_x + S + \lambda A$:

$$-\psi_x + (S + \lambda A)\psi = \mu\psi. \tag{4.5}$$

This latter problem can be solved via the observation that the integral leaves E are orbits of a group action, completely characterized by the distribution E at the special point B. There holds indeed:

Proposition 4.3. *The subspace $\mathfrak{g}_{AB} := \{V \in \mathfrak{g}_A | V_x + [V, B] \in \mathfrak{g}_A^{\perp}\}$ is a subalgebra of $\mathfrak{g}$ contained in the nilpotent subalgebra $\mathfrak{n}_-$ of loops with values in the maximal nilpotent subalgebra spanned by the negative roots. Therefore the corresponding group $G_{AB} = \exp(\mathfrak{g}_{AB})$ is well defined. The distribution E is spanned by the vector fields $(P_1)_B(V)$ with V belonging to $\mathfrak{g}_{AB}$, and its integral leaves are the orbits of the gauge action of G_{AB} on $\mathcal{S}$ defined by*

$$S' = TST^{-1} + T_xT^{-1}. \tag{4.6}$$

Now it easy to see that Eq. (4.6) implies that on the space $\mathrm{End}(C^\infty(S^1, \mathbb{C}^7))$ the linear differential operators $-\partial_x + S + \lambda A$ and $-\partial_x + S' + \lambda A$ are conjugated by an element $T \in G_{AB}$ in the formula

$$(-\partial_x + S' + \lambda A) \circ T = T \circ (-\partial_x + S + \lambda A) \tag{4.7}$$

if S and S' satisfy (4.6) with the same T.

Therefore if we define $v^{(0)} = (1, 0, 0, 0, 0, 0, 0)$ and by recurrence

$$v^{(j+1)}(S) = \partial_x v^{(j)}(S) + (S + \lambda A)v^{(j)}(S) \quad (v^{(0)}(S) = v^{(0)}) \tag{4.8}$$

then we have the

Proposition 4.4. *The vectors $v^{(j)}(S)$, $j \geq 0$, are covariant, i.e., $v^{(j)}(S') = v^{(j)}(S)T$, whenever $S' = TST^{-1} - T_xT^{-1}$ with $T \in G_{AB}$. Moreover the subset $\{v^{(j)}\}_{j=0,\ldots,6}$ is for any S in $\mathcal{S}$ a basis for $\mathbb{C}^7$.*

Developing now the first dependent vector, namely $v^{(7)}(S)$, we obtain the relation

$$\begin{aligned}v^{(7)}(S) &= 2u_0v^{(5)}(S) + 5u_0'v^{(4)}(S) + (6u_0'' - u_0^2)v^{(3)}(S) + (4u_0''' - 3u_0u_0')v^{(2)}(S)\\ &\quad + (24u_1 - 4\lambda - (u_0')^2 - u_0u_0'' + u_0^{(4)})v^{(1)}(S) + 12u_1'v^{(0)}(S)\end{aligned} \tag{4.9}$$

called the characteristic equation; moreover it is not difficult to show that, by construction, u_0 and u_1 are a complete set of invariants for the action of $\mathfrak{g}_{AB}$, i.e., they can be used to parameterize the quotient space $\mathcal{N}$.

These covariant vectors are the main tool for solving the eigenvector problem stated above. There holds namely

Proposition 4.5. *If ψ is the element of $C^\infty(S^1, \mathbb{C}^7)$ defined by the relations $\langle v^{(0)}, \psi\rangle = 1$, $\langle v^{(1)}, \psi\rangle = h$ and $\langle v^{(k)}, \psi\rangle = h^{(k)}$, $k = 2, \dots, 6$, where the functions $h^{(k)}$ are defined by the recurrence $h^{(1)} = h$, $h^{(k+1)} = h_x^{(k)} + h^{(k)}h$ and h satisfies the "Riccati"-type equation*

$$\begin{aligned} h^{(7)} &= 2u_0h^{(5)} + 5u_0'h^{(4)} + (6u_0'' - u_0^2)h^{(3)} + (4u_0''' - 3u_0u_0')h^{(2)} \\ &\quad + (24u_1 - 4\lambda - (u_0')^2 - u_0u_0'' + u_0^{(4)})h + 12u_1', \end{aligned} \tag{4.10}$$

then ψ is an eigenvector of $-\partial_x + S + \lambda A$ with eigenvalue $h(z)$. Moreover Eq. (4.10) admits a solution of the form $h(z) = cz + \sum_{i<0} h_i z^{-i}$ where $z^6 = \lambda$, and the coefficients h_k are obtained iteratively in an algebraic way.

Remark 4.6. Up to the transformation $h = \frac{\psi_x}{\psi}$ and the change of coordinates

$$u_0 = -u \qquad u_1 = \frac{1}{12}(u^{(4)} - u''u - (u')^2 - v)$$

the Eq. (4.10) coincides with the spectral problem arising from the Lax operator for $G_2^{(1)}$ given in [11].

The Laurent expansion can be effectively computed using (4.10); its first terms are

$$\begin{aligned} c &= 2^{1/3} \\ h_0 &= 0 \\ h_1 &= \frac{2^{1/3}}{6}u_0 \\ h_2 &= -\frac{2^{1/3}}{6}u_0' \\ h_3 &= \frac{1}{9}u_0'' \\ h_4 &= -\frac{2^{2/3}}{36}u_0''' \\ h_5 &= \frac{2^{1/3}}{108}\left(u_0''u_0 + u_0^{(4)} - \frac{u_0^3}{3} + 108u_1\right) \\ h_6 &= \dots. \end{aligned} \tag{4.11}$$

As expected, the coefficients of h corresponding to powers of z which are not $\pm 1 \bmod (6)$ are total derivatives. Moreover h_1 and h_5 are densities of the functionals (3.2) which are elements of the kernel of P_0. The Laurent series $h(z)$ plays the main role in our construction of the hierarchy related to $G_2^{(1)}$; we are indeed going to show that the knowledge of such a function together with the existence of a complete set of Casimirs of (4.1) is enough for computing all the soliton equations of the hierarchy. Let us indeed first use the invariance under the Weyl group of G_2 of the eigenvalues of the matrix $B + \lambda A$ (which belongs to a Cartan subalgebra of G_2; or the very expression of Eq. (4.10) where only $\lambda = z^6$ explicitly appears), sure that if $\psi(z)$ and $\mu(z)$ are respectively an eigenvector and an eigenvalue of the operator $-\partial_x + S + \lambda A$, then also $\psi_k(z) = \psi(\mathrm{e}^{\frac{2\pi i k}{6}} z)$ and $\mu_k(z) = \mu(\mathrm{e}^{\frac{2\pi i k}{6}} z)$ are for any $k = 0, \dots, 5$ another pair of respectively an eigenvector and an eigenvalue. Moreover it is easy to show that for any fixed x in S^1 the elements $\psi(\mathrm{e}^{\frac{2\pi i k}{6}} z)$, $k = 0, \dots, 5$, together with the obviously existing "zero"-eigenvector $\chi(z)$, form a basis of $\mathbb{C}^7$.

Further, using the expansion of V^j in Proposition 4.2 it easy to show that any flow of the hierarchy may be written as

$$\dot{S}_j = [A, V_j]. \tag{4.12}$$

Then since V^j is a solution of (4.1) we have that

$$((V^j)_+)_x + \left[((V^j)_+), S + \lambda A\right] = -((V^j)_-)_x - \left[((V^j)_-), S + \lambda A\right]$$

where $(\cdot)_+$ and $(\cdot)_-$ are respectively the projection on the regular and singular parts of the Laurent series V^j. This latter equation implies that

$$\left[A, V_j\right] = ((V^j)_+)_x + \left[((V^j)_+), S + \lambda A\right]$$

from which follows as usual, together with (4.12), the bi-Hamiltonian form of the equations of the hierarchy

$$\dot{S}_j = \left[A, V_j\right] = ((V^j)_+)_x + \left[(V^j)_+, S + \lambda A\right]. \tag{4.13}$$

It remains only to show how to rewrite these equations directly in terms of the function $h(x)$:

Proposition 4.7. *The Laurent series $h(x)$ evolves as*

$$\partial_{t_j} h = \partial_x H^{(j)}, \tag{4.14}$$

where the Laurent series $H^{(j)}$, called in the literature 'currents', are given by $H^{(j)} = \langle v^{(0)}, (V^j)_+ \psi_0 \rangle$.

Proof. From Eq. (4.5) with $\mu = h$ and (4.13) it follows that

$$(-\partial_x + S + \lambda A)\phi = h\phi - h_{t_j}\psi_0 \tag{4.15}$$

where

$$\phi = (-\partial_{t_j} + V^j)\psi_0. \tag{4.16}$$

Let us now decompose ϕ with respect to the basis $\chi, \psi_0, \ldots, \psi_5$: $\phi = c_6\chi + \sum_{a=0}^{5} c_a\psi_a$; the Eq. (4.15) implies

$$-c_{6x}\chi + \sum_{a=0}^{5}(-c_{ax} + c_a h_a)\psi_a = hc_6\chi + \sum_{a=0}^{5} hc_a\psi_a - h_{t_j}\psi_1$$

where $h_0 = h$ and therefore $-c_{6x} = c_6 h$ and $-c_{ax} = c_a h_a = c_a h$ for $a = 1, \ldots, 5$, but c_a being a Laurent series in z and h and $h_a - h$ $(a = 1, \ldots, 5)$ a series of maximal degree 1. These latter equations give $c_a = 0$ for $a = 1, \ldots, 6$. Hence $\phi = c_0\psi_0$ which, taking into account (4.16), the definition of $H^{(j)}$ and the normalization of ψ_0, implies $c_0 = H^{(j)}$. Therefore $\phi = H^{(j)}\psi_0$, which plugged into (4.15) yields $h_{t_j} = \partial_x H^{(j)}$. □

In the case of the Lie algebras of type A the most important property of the functions $H^{(j)}$ is that they can be actually computed without using directly the Casimir V^j giving a really powerful way to write down all the equations of the hierarchy using simply Eq. (4.14). Matters are however much more complicated for the other affine Lie algebras, impeding a straightforward generalization of the theory. The difficulty arises from the fact that Eq. (4.10) does not imply any longer that $\lambda = z^6$ belongs (in the space $\mathcal{L}$ of all Laurent polynomials in z) to the linear span generated by the Faà di Bruno polynomials $H_+ = \langle h^{(i)} \rangle_{i \in \mathbb{N}}$. The only property which seems to still survive in our setting is the observation:

Proposition 4.8. *The Laurent series of $H^{(j)}$ is given by*

$$H^{(j)} = (z)^j + \sum_{l \geq 1} H_l^j z^{-l}. \tag{4.17}$$

Proof. By the definition of $H^{(j)}$ we have $H^{(j)} = \langle v^{(0)}, (V^j)_+\psi_0 \rangle = \langle v^{(0)}, V^j\psi_0 \rangle - \langle v^{(0)}, (V^j)_-\psi_0 \rangle = z^j - \langle v^{(0)}, (V^j)_-\psi_0 \rangle$, since $V^1\psi_0 = z\psi_0$. Then since $(V^j)_- = V_1^j \lambda^{-1} + \ldots$ and from the definition of the $h^{(k)}$ and the Riccati equation it follows that $\psi_0 = (1, cz + O(1), (cz)^2 + O(z), \ldots)^{\mathrm{T}}$; we have that

$$-\langle v^{(0)}, (V^j)_- \psi_0 \rangle = \frac{H_1^j}{z} + \cdots .$$

Hence $H^{(j)}$ has the form (4.17). □

Actually we can make some fair guesses at how the theory should be modified in order to take into account at least the affine Lie algebras corresponding to the classical simple Lie algebras. For instance in the case of the affine Lie algebras $B_n^{(1)}$ the equations of the hierarchy have the form (4.14) where h is constrained by the requirement that $z^{2n}h$ is in the span of the positive Faà di Bruno polynomials and the currents $H^{(k)}$ may be directly computed as the projections on H_+ of z^n with respect to the decomposition $\mathcal{L} = H_+ \oplus H_-$ where $H_- = \langle z^j \rangle_{j<0}$, together with the further constraint that the odd currents are linear combinations of strictly positive Faà di Bruno polynomials. From these latter currents (in the case when $n = 3$), those corresponding to the Lie algebra $G_2^{(1)}$ should be obtained by performing a further reduction.

Acknowledgements

We would like to thank Marco Pedroni and Youjin Zhang for many useful discussions on the subject.

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